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Electronic Journal of Probability

Electron. J. Probab. **0** (2012), no. 0, 1–21.
ISSN: `1083-6489` DOI: `10.1214/EJP.vVOL-PID`

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Manel Kacem*† Véronique Maume-Deschamps‡

Abstract

We prove a self normalized central limit theorem for a new mixing class of processes introduced in Kacem *et al.* (2013). This class is larger than the classical strongly mixing processes and thus our result is more general than Peligrad and Shao's (1995) and Shi's (2000) ones. The fact that some conditionally independent processes satisfy this kind of mixing properties motivated our study. We investigate the weak consistency as well as the asymptotic normality of the estimator of the variance that we propose.

Keywords: Self normalized central limit theorem ; Mixing processes ; Variance estimation ; Invariance principle.
AMS MSC 2010: NA.
Submitted to EJP on , final version accepted on .

1 Introduction

It is well-known that asymptotic theorems allow to obtain approximations of various laws. One of these theorem, which is fundamental in probability theory, is the central limit theorem (C.L.T.).
In [15] we have considered processes of dependent random variables (r.v.s) that are conditionally independent given a factor. In particular, we have assumed that the conditioning is with respect to an unbounded memory of the factor. These processes are of interest for example in risk theory. Our main result consisted in deriving some mixing properties for such processes and proving a Central Limit Theorem for processes satisfying this new mixing property. We restate here the following definition of mixing provided in [15].
In the definition below $\| \|_a$ and $\| \|_b$ are norms on bounded functions (or on subspaces of bounded functions).

*Université de Lyon, Université Claude Bernard Lyon 1, Laboratoire SAF, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, 69007 Lyon, France. E-mail: `manelkacem7@gmail.com`
†Conservatoire National des Arts et Métiers, Département EFAB, EA 4603, 40 rue des jeûneurs, 75002 Paris, France
‡Université de Lyon, Université Claude Bernard Lyon 1, Institut Camille Jordan ICJ UMR 5208 CNRS, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, 69007 Lyon, France.
E-mail: `veronique.maume@univ-lyon1.fr` `http://isfa.univ-lyon1.fr/vmaume/`

Definition 1.1. *Let u, v be integers, a sequence $(X_n)_{n\in\mathbb{N}}$ of r.v.s is said to be $\eta^*_{(u,v)}$-mixing, if there exist a function $r \mapsto \eta^*(r)$ decreasing to 0 as r goes to infinity and a constant $C(u,v) > 0$ such that for any real valued bounded functions f and g and for any multi-indices satisfying the relation ($\star$):*

$$i_1 < \cdots < i_u \le i_u < i_u + r \le j_1 < \cdots < j_v \le j_v, \tag{$\star$}$$

we have

$$\sup |Cov(f(X_{i_1}, \dots, X_{i_u}), g(X_{j_1}, \dots, X_{j_v}))| \le C(u,v)\eta^*(r)\|f\|_a\|g\|_b, \tag{1.1}$$

where the supremum is taken over all the sequences $(i_1, \dots, i_u)$ and $(j_1, \dots, j_v)$ satisfying ($\star$) and $r \le j_1 - i_u$ is the gap of time between past and future.

Depending on the norms used in Definition 1.1, we have various kind of mixing (see [2]). For example, if $\|\ \|_a = \|\ \|_b = \|\ \|_\infty$, we shall say that the process is $\alpha^*_{(u,v)}$-mixing and we shall write $\alpha^*(r)$ instead of $\eta^*(r)$. If the process is $\alpha^*_{(u,v)}$-mixing for all u, v and if $\sup_{u,v\in\mathbb{N}} C(u,v) \le C < \infty$ then the process is α-mixing in the sense of [23] and we shall write $\alpha(r)$ instead of $\eta^*(r)$.

Remark 1.2. *Let $h : \mathbb{R}^u \to \mathbb{R}$, be an arbitrary function and denote his lipschitz modulus by $Lip(h) = \sup_{x\neq y} |h(x) - h(y)| \,/\, \|x - y\|_1$. If*

$$\sup |Cov(f(X_{i_1}, \dots, X_{i_u}), g(X_{j_1}, \dots, X_{j_v}))| \le C(u,v)\eta^*(r)Lip(f)Lip(g), \tag{1.2}$$

where $C(u,v) = (uLip(f) + vLip(g) + uvLip(f)Lip(g))$ and $\eta^*(r) \to 0$ as r goes to infinity, then the process is said λ-weakly dependent. For more details on this type of mixing and on its applications we refer to [5].

In the case of classical mixing sequences (see i.e. [13, 6]), mixing coefficients are uniform on u and v. This is not the case for mixing sequences satisfying our mixing condition (1.1). The usefulness of this kind of mixing coefficients has been shown in [15] where conditionally independent examples that satisfy (1.1) are given.

Mixing processes have very interesting asymptotic properties especially in the stationary case. Many authors have investigated the classical central limit theorem (C.L.T.) for the sum $S_n = \sum_{i=1}^n X_i$ where X_i are r.v.s satisfying mixing conditions (see for examples [5, 23, 14]).

Note that when the real variance in a C.L.T. is replaced by an estimate from the given data, then the estimator is called a self-normalizer and we say that a self normalized central limit theorem (S.N.C.L.T.) holds. While for independent and identically distributed r.v.s it is very simple to estimate the variance, this is not the case for dependent r.v.s for which we should construct an estimator which takes into account the dependence structure. The present paper aims at providing a self-normalizer and a S.N.C.L.T. for mixing sequences satisfying (1.1). For α-mixing sequences, [25] took over the class of estimator introduced in [19] for ρ-mixing sequences and showed that this class gives consistent estimators for strongly mixing sequences of r.v.s. For stationary sequences of associated r.v.s we refer to [20] in which the self normalizer arises from classical Bernstein-Block-Technique (see [1]). [7] dealt with random vectors of λ-weakly dependent sequences. They use non-overlapping blocks denoted by

$$\Delta_{i,m} = \frac{1}{\sqrt{m}} \sum_{j=(i-1)(m+\ell)+1}^{(i-1)(m+\ell)+m} X_j,$$

in the construction of their estimator. The fact that blocks are non-overlapping leads to choose two parameter m and ℓ while for Peligrad and Shao's (1995) estimator the blocks denoted by

$$\widetilde{\Delta}_{i,m} = \frac{1}{\sqrt{m}} \sum_{j=i+1}^{i+m} X_j,$$

are overlapping and only the parameter m must be chosen. In addition, choosing non-overlapping blocks enables [7] to provide not only an estimator of the variance but also an estimator of the variance of the estimator.

We have proved in [15] that conditionally independent processes of r.v.s $(X_n)_{n\in\mathbb{N}}$ given a factor $(V_1,\dots,V_n)$ satisfy $\eta^*_{(u,v)}$- mixing conditions. For such processes mixing coefficient may be exponentially bounded with respect to u and v (namely $C(u,v) \le K^{u+v}$ with $K>0$) where the mixing coefficient for λ-weakly dependent r.v.s is a polynomial function of u and v and of Lipschitz functions (see Remark 1.2). This is our main motivation for providing a S.N.C.L.T. for $\eta^*_{(u,v)}$- mixing processes.
In this work we provide a self-normalizer which is a modified version of the one obtained in [19]. As in [25] we adapt this estimator to our kind of mixing. We investigate its weak consistency as well as its asymptotic normality. In this frame we derive useful tools such as a covariance inequality of Davydov's kind and moment inequalities of Rosenthal's kind which are adapted with our structure of mixing.

This work is organized as follows. In Section 2 we give our main preliminary results: moment inequalities, covariance inequality and uniform integrability property. These are the main tools for the proofs of our results. In Section 3 we present our main results: we provide an estimator of the variance and we derive a S.N.C.L.T. (see Theorem 3.5). Also, the asymptotic normality of our estimator is proved (see Theorem 3.7). Section 4 is devoted to the proofs. In Section 5, we recall some examples that satisfy our assumptions. These examples may be relevant for example in risk theory contexts, they have already been presented in [15].

2 Main tools: moment inequalities, covariance inequality and uniform integrability

In all this paper, we shall use indifferently $\mathbb{E}(|f(X)|^p)^{\frac{1}{p}}$ or $\|f\|_p$ when there is no ambiguity.

2.1 Covariance inequality

In the following proposition we restate a covariance inequality which is adapted with our structure of mixing and which is proved in [15].

Proposition 2.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(u,v)}$-mixing sequence. Let $\|f\|_p<\infty$ and $\|g\|_q<\infty$, where $1<p,q\le\infty$ and $\frac{1}{p}+\frac{1}{q}<1$, then*

$$|Cov(f(X_{i_1},\dots,X_{i_u}),g(X_{j_1},\dots,X_{j_v}))| \le (C(u,v)+8)(\mathbb{E}|f|^p)^{\frac{1}{p}}(\mathbb{E}|g|^q)^{\frac{1}{q}}\alpha^*(r)^s, \qquad (2.1)$$

where $s=1-\frac{1}{p}-\frac{1}{q}$ and $j_v-i_u=r$ where r is a positive integer.

Remark 2.2. *Note that if $(X_n)_{n\in\mathbb{N}}$ is a sequence of complex valued r.v.s then by separating the real and imaginary parts of the complex number, we obtain inequality (2.1) with $4*(C(u,v)+8)$ instead of $(C(u,v)+8)$.*

2.2 Moment inequalities

Moment inequalities of partial sums play a very important role in various proofs of limit theorems. For independent r.v.s we have the following Rosenthal type inequality

$$\mathbb{E}|X_1+\ldots+X_n|^q \leq C_q(\sum_{i=1}^{n}\mathbb{E}|X_i|^q + (\sum_{i=1}^{n}\mathbb{E}|X_i|^2)^{q/2}). \tag{2.2}$$

In Lemmas 2.4 and 2.5 we provide Rosenthal type inequalities for mixing sequences that satisfy the $\alpha^*_{(u,v)}$-mixing condition defined by inequality (1.1). Recall the definition of the coefficients of weak dependence introduced in [8].

Definition 2.3. *For positive integer r, define the coefficients of weak dependence as the non-decreasing sequence $(C_{r,q})_{(q\geq 2)}$ such that*

$$C_{r,q} := \sup\left|\mathit{Cov}(X_{t_1}\times\ldots\times X_{t_m}, X_{t_{m+1}}\times\ldots\times X_{t_q})\right|,$$

where the supremum is taken over the multi-indices $1\leq t_1\leq\cdots\leq t_q$ with $t_{m+1}-t_m=r$.

We give the following lemma for bounded r.v.s satisfying the $\alpha^*_{(u,v)}$-mixing condition.

Lemma 2.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $\sup_{i\in\mathbb{N}}|X_i|\leq M$. In addition assume that $(X_n)_{n\in\mathbb{N}}$ is $\alpha^*_{(m,q-m)}$-mixing with $1\leq m<q$. Then for any positive integer r, we have*

$$C_{r,q}\leq M^q C(m,q-m)\alpha^*(r).$$

For non bounded r.v.s we derive the following lemma which is straightforward from Proposition 2.1 and Lemma 4.6 in [5].

Lemma 2.5. *Assume that the sequence of r.v.s $(X_n)_{n\in\mathbb{N}}$ is centered and let $S_n=\sum_{i=1}^{n}X_i$. If the sequence $(X_n)_{n\in\mathbb{N}}$ is $\alpha^*_{(1,1)}$-mixing and for some $\delta>0$, $\sup_{i\in\mathbb{N}}\|X_i\|_{2+\delta}<\infty$ then*

$$\mathbb{E}(S_n^2)\leq 2nK_2\sum_{r=0}^{n-1}\alpha^*(r)^{\frac{\delta}{\delta+2}}. \tag{2.3}$$

If the sequence $(X_n)_{n\in\mathbb{N}}$ is $\alpha^*_{(3,3)}$-mixing such that $\sup_{i\in\mathbb{N}}\|X_i\|_{4+\delta}<\infty$ for some $\delta>0$ then

$$\mathbb{E}(S_n^4)\leq 4!n^2K_2^2\left(\sum_{r=0}^{n-1}\alpha^*(r)^{\frac{\delta}{\delta+2}}\right)^2 + nK_4\sum_{r=0}^{n-1}(r+1)^2\alpha^*(r)^{\frac{\delta}{\delta+2}}, \tag{2.4}$$

where

$$K_2=(C(1,1)+8)\sup_{i\in\mathbb{N}}\|X_i\|^2_{2+\delta} \text{ and } K_4=(C(3,3)+8)\sup_{i\in\mathbb{N}}\|X_i\|^4_{4+2\delta}.$$

In Section 3 we need an estimation of the moments of the sum of order p greater than 2 where p is not necessary an integer. To this aim we prove the following theorem which apply to α^*-mixing processes. It is a generalized version of Theorem 4.1 in [24]. For $\delta>0$ fixed, $n\in\mathbb{N}^*$, $p\geq s>0$, we define

$$D_{n,s,p}=\left(\sum_{q=0}^{n-1}(1+q)^{p-s}\alpha^*(q)^{\frac{\delta}{\delta+2}}\right)^{\left(\frac{\delta}{\delta+s}+1\right)}. \tag{2.5}$$

Theorem 2.6. *Let $2 < p < r < \infty$, $2 < v \le r$ and $(X_n)_{n\in\mathbb{N}}$ be an $\alpha^*_{(m,2)}$-mixing sequence of r.v.s where $1 < m < [\frac{n}{2}] + 1$. Let $\mathbb{E}X_n = 0$ and assume for all $n \in \mathbb{N}$, $\|X_n\|_r < \infty$. Assume that for some $C > 0$ and $\theta > 0$*

$$\alpha^*(n) \le Cn^{-\theta}. \tag{2.6}$$

Then, for any $\epsilon > 0$ there exists $A = A(\varepsilon, r, p, \theta, C) < \infty$ such that

$$E|S_n|^p \le A\left(n^{p/2}(K_2 D_{n,2,2})^{p/2} + n^{(p-(r-p)\theta/r)\vee(1+\epsilon)} \sup_{i\le n} \|X_i\|_r^p\right). \tag{2.7}$$

In particular, for any $\varepsilon > 0$, if $\theta \ge (p-1)r/(r-p)$ then

$$E|S_n|^p \le A\left(n^{p/2}(K_2 D_{n,2,2})^{p/2} + n^{(1+\varepsilon)} \sup_{i\le n} \|X_i\|_r^p\right), \tag{2.8}$$

and if $\theta \ge \frac{pr}{(2(r-p))}$ then

$$E|S_n|^p \le An^{p/2}\left((K_2 D_{n,2,2})^{p/2} + \sup_{i\le n} \|X_i\|_r^p\right). \tag{2.9}$$

2.3 Uniform integrability

The concept of uniform integrability (U.I.) is essential in the proof of our limit theorems. Recall that if $\lim\limits_{A\to\infty} \sup\limits_{n\in\mathbb{N}} \mathbb{E}(|X_n| 1_{|X_n|>A}) = 0$ then the family of r.v.s $(X_n)_{n\in\mathbb{N}}$ is said uniformly integrable. Lemma 2.5 leads to the following corollary which gives sufficient conditions to have U.I. condition of order p where $p \in \{2, 4\}$.

Corollary 2.7. *Choose p an integer in $\{2, 4\}$. Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(p-1,p-1)}$ mixing sequence. Without loss of generality consider that $\mathbb{E}(X_i) = 0$ and assume that $\mathbb{E}(|X_i|^{p+\delta}) < \infty$ for some $\delta > 0$. If for each s in $\{2, p\}$*

$$D_{\infty,2,s} < \infty,$$

then

$$\left\{\left|\frac{S_n}{\sqrt{n}}\right|^p, n \ge 1\right\},$$

is uniformly integrable.

Remark 2.8. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(m,2)}$-mixing sequence of random variables, where $1 < m < [\frac{n}{2}] + 1$. Assume that for some, $C > 0$, $\delta > 0$, $p > 2$, $a > \frac{p(p+\delta)}{2\delta}$,*

$$\alpha^*(n) \le Cn^{-a},\ E|X_1|^{p+\delta} < \infty,$$

then Theorem 2.6 implies that

$$\left\{\left|\frac{S_n - \mathbb{E}(S_n)}{\sqrt{n}}\right|^p, n \ge 1\right\} \text{ is uniformly integrable.}$$

3 Self Normalized Central Limit Theorem for stationary strongly mixing sequences

In the literature, to prove limit theorems for strongly mixing sequences of r.v.s, one may approximate the strongly mixing sequence by another sequence. For examples, [9, 10] considers a direct approximation of mixing sequences by a sequence of martingale differences (for more details we refer to [26] and [11]). Another method known as

Bernstein's method (see [1]) is to consider a direct approximation of mixing sequences by a sequence of independent r.v.s. This is the method used in the proof of the central limit theorem derived in [13]. We refer to [3, 5, 6] for a survey on C.L.T. for mixing sequences.

Remark 3.1. *Assume that $(X_n)_{n\in\mathbb{N}}$ is a stationary centered sequence. If $\sum_{j=1}^{\infty}\mathbb{E}(X_0X_j)<\infty$ then*

$$\sigma^2 := \mathbb{E}(X_0^2)+2\sum_{j=1}^{\infty}\mathbb{E}(X_0X_j)<\infty. \tag{3.1}$$

If $\sigma>0$ then,

$$\sigma_n^2 := \mathbb{E}(\sum_{j=0}^{n}X_j)^2 = n\mathbb{E}(X_0^2)+2\sum_{j=1}^{n}(n-j)Cov(X_0,X_j)=n\sigma^2(1+o(1)). \tag{3.2}$$

Before stating a S.N.C.L.T. for $\alpha^*_{(u,v)}-$ mixing sequence of r.v.s we recall the following theorem proved in [15] for $\alpha^*_{(u,v)}$-mixing process.

Theorem 3.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(u,v)}$-mixing process for all $(u,v)\in\mathbb{N}^*$. Let $\mathbb{E}(X_1)=\mu$ and assume for a fixed $\delta>0$*

$$\mathbb{E}\,|X_1|^{2+\delta}<\infty \ \text{ and } \sum_{r=1}^{\infty}\alpha^*(r)^{\frac{\delta}{2+\delta}}<\infty. \tag{3.3}$$

If in addition there exist $M>0$ and $K>0$ such that for any u,v, and r,

$$\begin{cases} C(u,v)\le K^{u+v}, \\ \alpha^*(r)\le \frac{M}{r^a}, \quad a>\max\{2\ln(K)-1,2\}. \end{cases} \tag{3.4}$$

Then,

$$\frac{S_n-n\mu}{\sqrt{n}\sigma}\xrightarrow{\mathcal{L}} N(0,1). \tag{3.5}$$

Remark 3.3. *Condition (3.3) or (3.4) on α^* implies (3.1).*

The invariance principle has been extensively investigated for weakly dependent and mixing sequences (see i.e. [17, 18, 22, 4]). In particular [16] derived a weak invariance principle (W.I.P.) for dependent r.v.s which is extended by [12] for α-mixing sequence of r.v.s. In the following theorem we provide a W.I.P. for $\alpha^*_{(u,v)}$-mixing sequence. Let $\{W(t),0\le t\le 1\}$ be a standard Wiener process and define by $W_n(t)=(S_{[nt]}-[nt]\mu)/\sigma_n, 0\le t\le 1$.

Theorem 3.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(u,v)}$-mixing process for all $(u,v)\in\mathbb{N}^*\times\mathbb{N}^*$. Let $\mathbb{E}(X_1)=\mu$ and assume that (3.3) and (3.4) hold. Then,*

$$W_n\Rightarrow W, \tag{3.6}$$

where $\Rightarrow$ stands for the weak convergence.

Now we introduce the following estimator which is a modified version of some estimators introduced in [19] (we write $\ell_n=\ell$).

$$B_{n,2}^2=\frac{1}{n-\ell+1}\sum_{j=0}^{n-\ell}\left(\frac{S_j(\ell)-\overline{X}_\ell}{\sqrt{\ell}}\right)^2$$

and

$$\widehat{B}_{n,2}^2 = \frac{1}{n-\ell+1}\sum_{j=0}^{n-\ell}\left(\frac{S_j(\ell)-\ell\mu}{\sqrt{\ell}}\right)^2$$

with

$$S_j(\ell) = \sum_{k=j+1}^{j+\ell} X_k \text{ and } \overline{X}_\ell = \frac{1}{n-\ell+1}\sum_{j=0}^{n-\ell} S_j(\ell).$$

Note that as $n \to \infty$, (3.2) may be written as follows

$$\frac{\mathrm{Var}(S_n)}{n} := \frac{\sigma_n^2}{n} \to \sigma^2. \tag{3.7}$$

The following result proves that $B_{n,2}^2$ is a consistent estimator of σ^2.

Theorem 3.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary process wich is $\alpha^*_{(u,v)}$-mixing for all integers u and v. Assume that (3.3) holds. In addition assume that there exists $K > 0$ such that for all integers u and v,*

$$C(u,v) \le K^{u+v} \text{ and consider } \ell_n = \ell,\ \ell = [\ln(n)/5\ln(K)] + x, \tag{3.8}$$

where $[y]$ denotes the integer part of y and x is an integer. If $0 < \sigma < \infty$, then

$$B_{n,2} \to \sigma \text{ in } L^2. \tag{3.9}$$

Remark 3.6. *Note that a consequence of (3.5) and (3.9) is that*

$$\frac{S_n - n\mu}{\sqrt{n}B_{n,2}} \xrightarrow{\mathcal{L}} N(0,1). \tag{3.10}$$

This proves a S.N.C.L.T. for mixing processes satisfying our $\alpha^*_{(u,v)}-$ mixing condition.

Finally, we state the asymptotic normality of $B_{n,2}$.

Theorem 3.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(u,v)}$-mixing process for all integers u and v, with $\mathbb{E}(X_1) = \mu$. Denote by $S(\ell) = \sum_{k=1}^{\ell} X_k$ and let ℓ be as in (3.8). Assume (3.3), (3.7), it exists $\delta > 0$ such that $\mathbb{E}(|X_i|^{4+3\delta}) < \infty$ and that there exist $K > 0$, $C > 0$ such that for all integers u, v, r*

$$\begin{cases} C(u,v) \le K^{u+v}, \\ \alpha^*(r) \le \frac{C}{r^a}, \text{ with } a > \max\left\{\frac{(\delta+2)(4+3\delta)}{\delta}, 2\ln(K)-1\right\}, \end{cases} \tag{3.11}$$

Then,

$$\sqrt{\frac{n}{\ell}}\left(B_{n,2} - \mathbb{E}\left(\left|\frac{S(\ell)-\ell\mu}{\sqrt{\ell}}\right|^2\right)^{\frac{1}{2}}\right) \xrightarrow{\mathcal{L}} N(0, \frac{\sigma^2}{3}). \tag{3.12}$$

Remark 3.8. *Note that the choice of ℓ depends on the behavior of $C(u,v)$. Assume that there is $K > 0$ such that for all (u,v) we have*

1. *Case 1: $C(u,v) \le K$ we may choose $\ell_n = o(n)$ and Theorems 3.5 and 3.7 hold.*

2. *Case 2: $C(u,v) \le K(u+v)^\beta$ i.e. $C(u,v)$ is a polynomial on u and v then we can choose $\ell_n = n^s$ with $s < \frac{1}{4}$ if $\beta < 4$ and $s < \frac{1}{\beta+1}$ if $\beta \ge 4$ and Theorems 3.5 and 3.7 hold.*

4 Proof of the main results

Proof of Theorem 2.6. Because in our case the mixing coefficients depend on u and v, classical theorems such as Theorem 4.1 in [24] don't hold. We may nevertheless follow the lines of their proof. We shall prove inequality (2.7) by induction on n. Assume that

$$E|S_k|^p \leq A\left(k^{p/2}(K_2 D_{n,2,2})^{p/2} + k^{p+\frac{\theta(p-r)}{r}\vee(1+\epsilon)}\|X\|_r^p\right) \tag{4.1}$$

holds for each $1 < k < n$. Note that (4.1) is obvious for $k = 1$. We shall prove that (4.1) holds for $k = n$. Define

$$\xi_i = \sum_{j=1+2(i-1)m}^{n\wedge(2i-1)m} X_j \ , \ \eta_i = \sum_{j=1+(2i-1)m}^{n\wedge 2im} X_j, \tag{4.2}$$

where $1 \leq i \leq k_n := 1 + [n/(2m)]$. Note that for fixed i there exists a gap of length m between ξ_i and η_i. Let a be a constant such that $0 < a < \frac{1}{2}$ and take $m = [an] + 1$. By Minkowski inequality we write

$$E|S_n|^p \leq 2^{p-1}(\mathbb{E}|\sum_{i=1}^{k_n}\xi_i|^p + \mathbb{E}|\sum_{i=1}^{k_n}\eta_i|^p) := 2^{p-1}(I_1 + I_2). \tag{4.3}$$

The proof of Theorem 2.6 requires the following Lemma due to [24].

Lemma 4.1 ([24], Lemma 4.1)**.** *Let $\xi_i, 1 \leq i \leq n$ be a sequence of r.v.s and let F_i be the σ-field generated by $(\xi_j, j \leq i)$. Then for any $p \geq 2$ there is a constant $D = D(p)$ such that*

$$\mathbb{E}|\sum_{i=1}^n \xi_i|^p \leq D \times [(\sum_{i=1}^n \mathbb{E}(\xi_i^2))^{\frac{p}{2}} + \sum_{i=1}^n \mathbb{E}|\xi_i|^p + n^{p-1}\sum_{i=1}^n \mathbb{E}|\mathbb{E}(\xi_i|F_{i-1})^p +$$
$$n^{\frac{p}{2}-1} \times |\sum_{i=1}^n \mathbb{E}|\mathbb{E}(\xi_i^2|F_{i-1}) - \mathbb{E}(\xi_i^2)|^{\frac{p}{2}}].$$

In order to simplify the notations, in what follows, $\|X\|_r$ stands for $\sup_{i\leq n}\|X_i\|_r$.

Replacing n by k_n in Lemma 4.1 yields to

$$I_1 \leq D \times [\underbrace{(\sum_{i=1}^{k_n}\mathbb{E}|\xi_i|^2)^{\frac{p}{2}}}_{I_{1,1}} + \sum_{i=1}^{k_n}\mathbb{E}|\xi_i|^p + \underbrace{k_n^{p-1}\sum_{i=1}^{k_n}\mathbb{E}|\mathbb{E}(\xi_i|F_{i-1})|^p}_{I_{1,2}} +$$
$$\underbrace{k_n^{\frac{p}{2}-1}\sum_{i=1}^{k_n}\mathbb{E}|\mathbb{E}(\xi_i^2|F_{i-1}) - \mathbb{E}(\xi_i^2)|^{\frac{p}{2}}}_{I_{1,3}}].$$

We begin with the bound for $I_{1,1}$. From (2.3) and (2.5) we deduce that

$$I_{1,1} \leq (k_n 2m K_2 D_{m,2,2})^{\frac{p}{2}} \leq 2^{\frac{p}{2}}(k_n m)^{\frac{p}{2}}(K_2 D_{m,2,2})^{\frac{p}{2}} \leq 2^p K_2^{\frac{p}{2}}(nD_{n,2,2})^{\frac{p}{2}}$$

where $k_n m < 2n$.
Now we consider $I_{1,3}$. Define $Y_i = \mathbb{E}(\xi_i^2|F_{i-1}) - \mathbb{E}(\xi_i^2)$ and denote by $F_{i-1} = \sigma(\xi_1, \ldots, \xi_{i-1})$.

Y_i is a measurable function with respect to F_{i-1}. Write $\mathbb{E}(|Y_i|^{p/2}) = \mathbb{E}|Y_i|^{(p/2)-1}sgn(Y_i)Y_i$ then we have

$$\begin{aligned}\mathbb{E}(|Y_i|^{p/2}) &= \mathbb{E}(|Y_i|^{\frac{p}{2}-1}\text{sgn}(Y_i)(\xi_i^2 - \mathbb{E}(\xi_i^2))) \\ &\leq \sup_{j,l} \sum_{2(i-1)m<j,l\leq n\wedge(2i-1)m} Cov(|Y_i|^{\frac{p}{2}-1}, X_jX_l - \mathbb{E}(X_jX_l)).\end{aligned}$$

Consider that

$$f(X_{1+2(i-1)m}, \dots, X_{n\wedge(2i-1)m}) = |Y_i|^{p/2-1} \text{ and } g(X_j, X_l) = |X_jX_l - \mathbb{E}(X_jX_l)|.$$

Choose p in (2.1) equals to $\frac{p}{p-2}$ and $q = \frac{r}{2}$ such that $\frac{2}{r} + \frac{p-2}{p} < 1$, in this case by Proposition 2.1 we have

$$\left|\text{Cov}\left(|Y_i|^{p/2-1}, X_jX_l - \mathbb{E}(X_jX_l)\right)\right| \leq (C(m,2)+8)(\mathbb{E}|Y_i|^{\frac{p}{2}})^{\frac{p-2}{p}} \|X_jX_l\|_{\frac{r}{2}} \alpha^*(m)^{\frac{2}{p}-\frac{2}{r}}.$$

Then

$$\mathbb{E}|Y_i|^{p/2} \leq (C(m,2)+8)m^2(\mathbb{E}|Y_i|^{\frac{p}{2}})^{\frac{p-2}{p}} \|X\|_r^2 \alpha^*(m)^{\frac{2}{p}-\frac{2}{r}}.$$

Hence

$$\mathbb{E}|Y_i|^{p/2} \leq (C(m,2)+8)^{\frac{p}{2}} m^p \|X\|_r^p \alpha^*(m)^{1-\frac{p}{r}}.$$

Let B be a constant. By inequality (2.6) and as $\theta > 0$ and $p < r$ we write

$$\begin{aligned}I_{1,3} &\leq k_n^{\frac{p}{2}}(C(m,2)+8)^{\frac{p}{2}} m^p \|X\|_r^p \alpha^*(m)^{1-\frac{p}{r}} \\ &\leq 2^p B(C(m,2)+8)^p n^{(p+\theta\frac{(p-r)}{r})\vee(1+\varepsilon)} a^{\theta\frac{(p-r)}{r}} \|X\|_r^p\end{aligned}$$

Now, let us focus on $I_{1,2}$. Define $Z_i = \mathbb{E}(\xi_i|F_{i-1})$ and denote by

$$f(X_{1+2(i-1)m}, \dots, X_{n\wedge(2i-1)m}) = |Y_i|^p \text{ and } g(X_j) = |X_j - \mathbb{E}(X_j)|.$$

Now choose p in (2.1) equals to $\frac{p}{p-1}$ and q equals to r where $\frac{1}{r} + \frac{p-1}{p} < 1$. Then in this case we have

$$\begin{aligned}\mathbb{E}(|Z_i|^p) = \mathbb{E}(|Z_i|^{p-1}\text{sgn}(Z_i)(\xi_i)) &\leq \sup \sum_{2(i-1)m<j\leq n\wedge(2i-1)m} Cov[(|Z_i|^{p-1}, X_j)] \\ &\leq (C(m,1)+8)m(\mathbb{E}|Z_i|^p)^{\frac{p-1}{p}} \|X\|_r \alpha^*(m)^{\frac{1}{p}-\frac{1}{r}}.\end{aligned}$$

Hence,

$$\mathbb{E}(|Z_i|^p) \leq (C(m,1)+8)^p m^p \|X\|_r^p \alpha^*(m)^{1-\frac{p}{r}}.$$

By inequality (2.6) and as $\theta > 0$ and $p < r$ we have

$$I_{1,2} \leq 2^p C(C(m,2)+8)^p n^{(p+\theta\frac{(p-r)}{r})\vee(1+\varepsilon)} a^{\theta\frac{(p-r)}{r}} \|X\|_r^p,$$

where C is a constant, whence

$$\begin{aligned}I_1 \leq D[&\sum_{i=1}^{k_n} \mathbb{E}|\xi_i|^p + 2^p K_2^{\frac{p}{2}}(nD_{n,2,2})^{\frac{p}{2}} + \\ &2^{p+1}B(C(m,2)+8)^p n^{(p+\theta\frac{(p-r)}{r})\vee(1+\varepsilon)} a^{\theta\frac{(p-r)}{r}} \|X\|_r^p].\end{aligned} \tag{4.4}$$

Similarly for

$$\begin{aligned}I_2 \leq D[&\sum_{i=1}^{k_n} \mathbb{E}|\eta_i|^p + 2^p K_2^{\frac{p}{2}}(nD_{n,2,2})^{\frac{p}{2}} \\ &+ 2^{p+1}B(C(m,2)+8)^p n^{(p+\theta\frac{(p-r)}{r})\vee(1+\varepsilon)} a^{\theta\frac{(p-r)}{r}} \|X\|_r^p].\end{aligned}$$

Consequently, we have

$$\mathbb{E}\,|S_n|^p \leq 2^{p-1}D[\sum_{i=1}^{k_n}\mathbb{E}|\xi_i|^p + \sum_{i=1}^{k_n}\mathbb{E}|\eta_i|^p + 2^{p+1}K_2^{\frac{p}{2}}(nD_{n,2,2})^{\frac{p}{2}} + 2B(C(m,2)+8)^p 2^{p+1} n^{(p+\theta\frac{(p-r)}{r})\vee(1+\varepsilon)} a^{\theta\frac{(p-r)}{r}}\,\|X\|_r^p].$$

Remark that $\sum_{i=1}^{k_n}\mathbb{E}|\xi_i|^p + \sum_{i=1}^{k_n}\mathbb{E}|\eta_i|^p \leq 2k_n\mathbb{E}|\xi_1|^p = 2k_n\mathbb{E}|S_m|^p$.
Take $a = (2^{p+4}D)^{-\frac{1}{\varepsilon}-\frac{2}{p-2}}$. Assume that (4.1) is true for all $k < n$ where

$$A = 2^{p+1}D\left[2^p + 2^{p+1}C\,(C(m,2)+8)^p\,a^{\theta\frac{(p-r)}{r}}\right].$$

In this case by (4.3), (4.4) and (4) we get

$$\begin{aligned}\mathbb{E}\,|S_n|^p &\leq 2^p D k_n A\left[m^{p/2}(K_2D_{m,2,2})^{p/2} + m^{p+\theta\frac{(p-r)}{r}\vee(1+\varepsilon)}\,\|X\|_r^p\right]\\ &+ \frac{A}{2}\left(n^{\frac{p}{2}}(K_2D_{n,2,2})^{\frac{p}{2}} + n^{(p+\theta\frac{(p-r)}{r})\vee(1+\epsilon)}\,\|X\|_r^p\right)\\ &\leq A\left[n^{p/2}(K_2D_{n,2,2})^{p/2} + n^{(p+\frac{\theta(p-r)}{r})\vee(1+\varepsilon)}\,\|X\|_r^p\right].\end{aligned} \tag{4.5}$$

So that, finally (4.1) is valid for $k = n$. $\square$

Before proceeding to the proof of Theorem 3.5, we give the following lemma which is useful in the proof of this theorem. This lemma is a variant of Lemma 2.3 in [25].

Lemma 4.2. *Let $(X_n, n \geq 1)$ be $\alpha^*_{(\ell_n,\ell_n)}$-mixing sequence where $1 \leq \ell_n \leq n$. Let f be a real function on $\mathbb{R}^{\ell_n}$ and put $Z_i = f(X_{i+1},\ldots,X_{i+\ell_n})$. Assume that $\mathbb{E}\,|Z_k|^{2+\delta} < \infty$ for some $\delta > 0$ and $\sup_{j\in\mathbb{N}}\|Z_j\|_{2+\delta}^2 < \infty$. Then we have*

$$Var(\sum_{k=1}^{n} Z_k) \leq 2nK\ell_n \sup_{j\in\mathbb{N}}\|Z_j\|_{2+\delta}^2\,(C(\ell_n,\ell_n)+8)\sum_{r=1}^{[\frac{n}{\ell_n}]+1}\alpha^*(r)^{\frac{\delta}{\delta+2}}.$$

where K is a constant.

Proof of Lemma 4.2. Let f be a real function on $\mathbb{R}^{\ell_n}$ and denote by $Z_i = f(X_{i+1},\ldots,X_{i+\ell_n})$. Write

$$\mathrm{Var}(\sum_{k=1}^{n} Z_k) \leq \ell_n^2 \max_j \mathrm{Var}(\sum_{i=0}^{[\frac{n}{\ell_n}]} Z_{i\ell_n+j})$$

and use Proposition 2.1. $\square$

Proof of Theorem 3.4. To prove this theorem we follow the proof of the W.I.P. provided by [12] for $\alpha-$ mixing sequences. Note that the proof of the Theorem of [12] is done in a simplified version in [3]. Following their proof, the W.I.P. holds for $\alpha-$ mixing sequences if a C.L.T. holds and if tightness is verified. We note that a sufficient conditions for tightness is the following: for any $\epsilon > 0$, $\eta > 0$, there exist a δ, $0 < \delta < 1$, and an integer n_0 such that, for $0 \leq t \leq 1$,

$$\frac{1}{\delta}\mathbb{P}\left\{\sup_{t\leq s\leq s+\delta}|W_n(s) - W_n(t)| \geq \epsilon\right\} \leq \eta, \quad n \geq n_0. \tag{4.6}$$

Recall that a C.L.T. (Theorem 3.2) holds for $\alpha^*_{(u,v)}-$ mixing sequences, so that, the key to prove Theorem 3.4 lies in verification of (4.6). A careful analysis of the proof of [3] shows that if in addition to (3.3) and (3.4), there exists a positive integer $p = o(n)$ such that as $n \to \infty$,

$$(n/p) \max_{0\le m\le n-p} \mathbb{P}\left\{ \max_{1\le r\le p} |S_{m+r} - S_m| > \epsilon\sqrt{n} \right\} \to 0 \ \ for \ \ any \ \ \epsilon > 0, \tag{4.7}$$

then the tightness is verified. Equation (4.7) is implied by

$$\max_{0\le m\le n-p} \mathbb{E}\left(\left(\frac{\sum_{i=m+1}^{m+p} |X_i|}{\sqrt{p}} \right)^2 I\left(\frac{\sum_{i=m+1}^{m+p} |X_i|}{\sqrt{p}} > \epsilon\sqrt{\frac{n}{p}} \right) \right) \to 0 \text{ as } n \to \infty, \tag{4.8}$$

and (4.8) is satisfied provided that $\left\{ |\sum_{i=m+1}^{m+p} X_i/\sqrt{p}|^2, p \ge 1 \right\}$ is uniformly integrable which follows from Corollary 2.7. The proof is completed. □

Proof of Theorem 3.5. Without loss of generality consider that $\mu = 0$. We have $\sum_{n=1}^{\infty} \alpha^*(n)^{\frac{\delta}{2+\delta}} < \infty$ then, for large n, $\alpha^*(n)^{\frac{\delta}{\delta+2}} \le \frac{1}{n}$ whence $\alpha^*(n)n^{\frac{2}{\delta}} \le \alpha^*(n)^{\frac{\delta}{\delta+2}}$. First we prove that

$$|B_{n,2} - \widehat{B}_{n,2}| \to 0 \ \ in \ L^2 \ . \tag{4.9}$$

By Minkowski inequality we have

$$(\sum_{j=0}^{n-\ell} |\frac{S_j(\ell) - \overline{X}_\ell}{\sqrt{\ell}}|^2)^{\frac{1}{2}} \le (\sum_{j=0}^{n-\ell} |\frac{S_j(\ell)}{\sqrt{\ell}}|^2)^{\frac{1}{2}} + (\sum_{j=0}^{n-\ell} |\frac{\overline{X}_\ell}{\sqrt{\ell}}|^2)^{\frac{1}{2}}.$$

Hence,

$$|B_{n,2} - \widehat{B}_{n,2}| \le \frac{1}{\sqrt{n-\ell+1}} \times [(\sum_{j=0}^{n-\ell} |\frac{S_j(\ell)}{\sqrt{\ell}}|^2)^{\frac{1}{2}} + (\sum_{j=0}^{n-\ell} |\frac{\overline{X}_\ell}{\sqrt{\ell}}|^2)^{\frac{1}{2}} - (\sum_{j=0}^{n-\ell} |\frac{S_j(\ell)}{\sqrt{\ell}}|^2)^{\frac{1}{2}}].$$

We obtain

$$\mathbb{E}|B_{n,2} - \widehat{B}_{n,2}|^2 \le \frac{1}{n-\ell+1} \mathbb{E}|(\sum_{j=0}^{n-\ell} |\frac{\overline{X}_\ell}{\sqrt{\ell}}|^2)^{\frac{1}{2}}|^2 = \mathbb{E}|\frac{\overline{X}_\ell^2}{\ell}|.$$

Moreover,

$$\mathbb{E}|\frac{\overline{X}_\ell^2}{\ell}| = \frac{1}{\ell}\text{Var}(\overline{X}_\ell) = \frac{1}{\ell(n-\ell+1)^2}\text{Var}(\sum_{j=0}^{n-\ell} S_j(\ell)).$$

We write $S(\ell) = S_0(\ell)$. From Lemma 4.2

$$\frac{1}{\ell(n-\ell+1)^2}\text{Var}(\sum_{j=0}^{n-\ell} S_j(\ell)) \le \frac{2A'(C(\ell,\ell)+8)}{(n-\ell+1)} \|S(\ell)\|_{2+\delta}^2 \sum_{r=0}^{[\frac{n-\ell}{\ell}]+1} \alpha^*(r)^{\frac{\delta}{\delta+2}}.$$

where A' is a constant. The choice of ℓ_n depends on the behavior of $C(u,v)$ with respect to u and v. Since we assume that

$$C(u,v) \le K^{u+v}, \ \ \ell = [\frac{\ln(n)}{5\ln(K)}] + x, \tag{4.10}$$

we get (using Theorem 2.6):

$$\frac{2A'}{(n-\ell+1)} \sum_{r=0}^{[\frac{n-\ell}{\ell}]+1} \alpha^*(r)^{\frac{\delta}{\delta+2}} (K^{2x} n^{\frac{2}{5}} + 8) \|S(\ell)\|_{2+\delta}^2 = O(\frac{\ell}{n^{\frac{3}{5}}}) = o(1).$$

Hence (4.9) holds. In order to prove that $B_{n,2} \to \sigma$ as $n \to \infty$ it suffices to prove that

$$\widehat{B}_{n,2} \to \sigma \text{ in } L^2 \text{ as } n \to \infty. \tag{4.11}$$

Now we first prove that

$$\mathbb{E}|\widehat{B}_{n,2}^2 - \mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2| \to 0 \text{ as } n \to \infty. \tag{4.12}$$

Define

$$Z_{j,1} = (\frac{S_j(\ell)}{\sqrt{\ell}})^2 I(\frac{|S_j(\ell)|}{\sqrt{\ell}} \le (\frac{n}{\ell})^{1/8}) \text{ and } Z_{j,2} = (\frac{S_j(\ell)}{\sqrt{\ell}})^2 I(\frac{|S_j(\ell)|}{\sqrt{\ell}} > (\frac{n}{\ell})^{1/8}).$$

We have

$$\begin{aligned}
\mathbb{E}|\widehat{B}_{n,2}^2 - \mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2| &= \frac{1}{n-\ell+1}\mathbb{E}|\sum_{j=0}^{n-\ell}(Z_{j,1} - \mathbb{E}(Z_{j,1})) + \sum_{j=0}^{n-\ell}(Z_{j,2} - \mathbb{E}(Z_{j,2}))| \\
&\le K' n^{-1}(\mathbb{E}|\sum_{j=0}^{n-\ell} Z_{j,1} - \mathbb{E}(Z_{j,1})| + n\mathbb{E}|Z_{0,2}|) \\
&\le K'[(n^{-2}\mathrm{Var}(\sum_{j=0}^{n-\ell} Z_{j,1}))^{\frac{1}{2}} + \mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2 I(\frac{|S(\ell)|}{\sqrt{\ell}} > (\frac{n}{\ell})^{1/8})].
\end{aligned}$$

where K' is a constant. From Lemma 4.2 we obtain

$$n^{-2}\mathrm{Var}(\sum_{j=0}^{n-\ell} Z_{j,1}) \le \frac{2A_1'\ell(n-\ell+1)}{n^2}(C(\ell,\ell)+8)\,\|Z_{j,1}\|_{2+\delta}^2 \sum_{r=0}^{[\frac{n-\ell}{\ell}]+1} \alpha^*(r)^{\frac{\delta}{\delta+2}}$$

where A_1' is a constant. With our choice of ℓ:

$$\mathbb{E}|\widehat{B}_{n,2}^2 - \mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2| = O(\frac{\ln(n)}{n^{\frac{3}{5}}}).$$

On the other hand, the uniform integrability of $\left(S(\ell)/\sqrt{\ell}\right)^2$ implies that

$$\mathbb{E}[(\frac{S(\ell)}{\sqrt{\ell}})^2 I_{|\frac{S(\ell)}{\sqrt{\ell}}| > (\frac{n}{\ell})^{1/8}}] \to 0.$$

Then (4.12) holds. Moreover, the uniform integrability of $\left(S(\ell)/\sqrt{\ell}\right)^2$ implies

$$\mathbb{E}[(\frac{S(\ell)}{\sqrt{\ell}})^2 I_{|\frac{S(\ell)}{\sqrt{\ell}}| \le R}] \to \mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2, \text{ as } R \to \infty$$

and recall that from Remark 3.1

$$\mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2 \to \sigma^2 \text{ as } \ell \to \infty. \tag{4.13}$$

So that (4.12) implies

$$\mathbb{E}|\widehat{B}_{n,2}^2 - \sigma^2| \to 0 \text{ as } n \to \infty. \tag{4.14}$$

Finally, for any $x, y \ge 0$, for any $p \ge 1$.

$$|x-y|^2 \le K'|x^p - y^p|^{\frac{2}{p}},$$

so that (4.11) holds. As a conclusion, by (4.9) and (4.11) we have $B_{n,2} \to \sigma$ in L^2 as $n \to \infty$. □

Remark 4.3. *Note that if $C(u,v)$ is a polynomial function of u and v such that $C(u,v) \leq K(u+v)^{\beta}$ and if we choose $\ell = n^s$ with $s < \frac{1}{(1+\beta)}$ then Theorem 3.5 holds with the same proof.*

In the proof of Theorem 3.7 we need the following lemma on triangular arrays.

Lemma 4.4. *Let $(X_n)_{n\geq 1}$ be $\alpha^*_{(u,v)}$-mixing sequence of r.v.s for every $1 \leq u \leq v \leq n$. Assume that (3.3) and and 3.4 hold. Let $\{a_{nk;1\leq k\leq n}\}$ be a triangular array of real numbers such that*

$$\sup_n \sum_{k=1}^{n} a_{nk}^2 < \infty \text{ and } \sup |a_{nk}| \to 0 \text{ as } n \to \infty. \tag{4.15}$$

Assume that $\left\{|X_k|^{2+\delta}\right\}$ is a uniformly integrable family and that

$$\mathit{Var}(\sum_{k=1}^{n} a_{nk}X_k) \to \sigma^2.$$

Then

$$\frac{\sum_{k=1}^{n} a_{nk}X_k}{\sigma} \xrightarrow{\mathcal{L}} N(0,1) \tag{4.16}$$

as $n \to \infty$.

In order to prove Lemma 4.4 we need the following lemma whose proof follows the proof of Lemma 3.2 in [21].

Lemma 4.5. *Let $(X_n)_{n\in\mathbb{N}}$ be an $\alpha^*_{(1,1)}$-mixing sequence with $\sup_{k\in\mathbb{N}} \mathbb{E}\,|X_k|^{2+\delta} < \infty$ for some $\delta > 0$. Let $\{a_{nk}; 1 \leq k \leq n\}$ be a triangular array of real number such that*

$$\sum_{k=1}^{\infty} a_{nk}^2 < \infty \text{ and } \sup_{1\leq k\leq n} |a_{nk}| \to 0 \text{ as } n \to \infty. \tag{4.17}$$

Assume that (3.3) holds then for every $0 \leq a < b \leq n$

$$\mathit{Var}(\sum_{k=a}^{b} a_{nk}X_k) \leq C_1 \sum_{k=a}^{b} a_{nk}^2 < \infty, \tag{4.18}$$

where $C_1 = 2(C(1,1)+8)\sup_{i\in\mathbb{N}} \|X_i\|_{2+\delta}^2 \sum_{r=0} \alpha^*(r)^{\frac{\delta}{\delta+2}}$.

Proof of Lemma 4.5. Proceed as in [21] to get

$$\text{Var}(\sum_{k=a}^{b} a_{nk}X_k) \leq 2\sum_{k=a}^{b} a_{nk}^2 \sum_{k,j=a}^{b} |\text{Cov}(X_k, X_j)|$$

and use Lemma 2.5. □

Remark 4.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary centered sequence and let $\{a_{nk;1\leq k\leq n}\}$ be a triangular array of real numbers.*
If for any $k \geq 1$,

$$\sum_{t=1}^{\infty} a_{nt}\mathbb{E}(X_tX_k) < \infty$$

then

$$s_k^2 := \sum_{t=1}^{\infty} a_{nk}a_{nt}\mathbb{E}(X_tX_k) < \infty, \tag{4.19}$$

and

$$\sigma_n^2 := \mathbb{E}\left(\sum_{k=1}^n a_{nk}X_k\right)^2 = \sum_{k=1}^n s_k^2. \tag{4.20}$$

Clearly, we have

$$n \inf_{1\le k\le n} s_k^2 < \sigma_n^2 < n \sup_{1\le k\le n} s_k^2.$$

Now we prove Lemma 4.4

Proof of Lemma 4.4. We use a truncation technic. Let N be a constant, we write

$$X_i^{'} = X_i I(|X_i| \le N) - \mathbb{E}(X_i I(|X_i| \le N)),$$

$$X_i^{''} = X_i I(|X_i| > N) - \mathbb{E}(X_i I(|X_i| > N)),$$

From Lemma 4.5 and by the uniform integrability condition of $|X_i|^{2+\delta}$ we have

$$\lim_{N\to\infty} \sup_n \text{Var}(\sum_{i=1}^n a_{nk}X_{ni}^{''}) = 0.$$

so to prove this lemma it suffices to prove that

$$\frac{\sum_{i=1}^n a_{nk}X_i'}{\sqrt{\text{Var}(\sum_{i=1}^n a_{nk}X_i')}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1),$$

We follow the proof of Theorem 3.2 given in [15] which is based on Bernstein's method. We note that Lemma 4.4 holds if conditions (4.21) and (4.22) below are satisfied for some $p = o(n), q = [n/p]$ and $k = [n/(p+q)]$.

$$\lim_{n\to\infty} \frac{n\sup_{1\le k\le n} a_{nk}^2}{p\sigma_n^2} \int_{|Z|>\epsilon\sigma_n} z^2 dF_p(z) = 0, \tag{4.21}$$

with F_p the distribution function $F_p(z) = P(a_{n1}X_1' + \ldots + a_{np}X_p' < z)$ and

$$\lim_{n\to\infty} \mathbb{E}(\frac{1}{\sigma_n}\sum_{i=0}^k \eta_i)^2 = 0, \text{ where } \eta_i = \sum_{(i+1)p+iq+1}^{(i+1)p+(i+1)q} a_{nj}X_j' \text{ , } (0 \le i \le k-1). \tag{4.22}$$

Using Markov's inequality, we have

$$\frac{n\sup_{1\le k\le n} a_{nk}^2}{p\sigma_n^2} \int_{|Z|>\epsilon\sigma_n} z^2 dF_p(z) \le \frac{n\sup_{1\le k\le n} a_{nk}^4}{\epsilon^2 p\sigma_n^4}\mathbb{E}(\sum_{i=1}^p X_i')^4, \tag{4.23}$$

Assume without loss of generality that $\mathbb{E}(X_i') = 0$. Using that $\sum_{j=1}^\infty \alpha^*(j) < \infty$, the inequality $|\mathbb{E}(X_0', X_j')| \le C(1,1)N^2\alpha^*(j)$ and following Remark 4.6 we get

$$\lim_{n\to\infty} \frac{n\sup_{1\le k\le n} a_{nk}^2}{p\sigma_n^2} \int_{|Z|>\epsilon\sigma_n} z^2 dF_p(z) \le \lim_{n\to\infty} \frac{n\sup_{1\le k\le n} a_{nk}^4 E(\sum_{i=1}^p X_i')^4}{\epsilon^2 \inf_{1\le k\le n} s_k^4 pn^2}. \tag{4.24}$$

Lemma 4.6 in [5] implies that $E(\sum_{i=1}^n X_i')^4 = O(n^2)$ if

$\sum_{i=1}^\infty j\alpha^*(j) < \infty$. Hence the right hand of (4.23) is $O(\frac{p}{n})$ and goes to zero provided that $p = o(n)$. For the second condition (4.22) we have

$$\lim_{n\to\infty} \mathbb{E}[(\frac{1}{\sigma_n}\sum_{i=0}^k \eta_i)^2] \le \lim_{n\to\infty} \frac{\sup_{1\le k\le n} a_{nk}^2}{\inf_{1\le k\le n} s_k^2 n}[\sum_{i=1}^k \mathbb{E}(\eta_i^2) + 2\sum_{i<j\le k} \mathbb{E}(\eta_i\eta_j)]. \tag{4.25}$$

Following the proof of Theorem 3.2 in [15], we get Lemma 4.4. □

Theorem 3.7 follows from the proposition below which is the same result as in Theorem 3.7 but for $\widehat{B}_{n,2}$ instead of $B_{n,2}$.

Proposition 4.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\alpha^*_{(u,v)}$-mixing process for all $(u,v)\in\mathbb{N}^*$ with $\mathbb{E}(X_1)=\mu$. Denote by $S(\ell)=\sum_{k=1}^{\ell}X_k$ and let ℓ be as in (3.8). Assume (3.3), (3.7), (3.11) and $\mathbb{E}(|X_i|^{4+3\delta})<\infty$. Then,*

$$\sqrt{\frac{n}{\ell}}(\widehat{B}_{n,2}-\mathbb{E}(|\frac{S(\ell)-\ell\mu}{\sqrt{\ell}}|^2)^{\frac12})\xrightarrow{\mathcal{L}}N(0,\frac{\sigma^2}{3}). \tag{4.26}$$

Proof of Proposition 4.7. Assume without loss of generality that $\mathbb{E}(X_1)=0$. Let $r=o(n)$ and $\ell=o(r)$. Define

$$\xi_{m,n}=\sum_{j=m(2\ell+r)}^{m(2\ell+r)+r-1}[(\frac{S_j(\ell)}{\sqrt{\ell}})^2-\mathbb{E}(\frac{S_j(\ell)}{\sqrt{\ell}})^2],$$

$$\eta_{m,n}=\sum_{j=m(2\ell+r)+r}^{(m+1)(2\ell+r)-1}[(\frac{S_j(\ell)}{\sqrt{\ell}})^2-\mathbb{E}(\frac{S_j(\ell)}{\sqrt{\ell}})^2],$$

where $m=0,1,\ldots,k_n:=[(n-\ell+1)/(2\ell+r)]-1$, ξ is a partial sum of r terms and η is a partial sum of 2ℓ terms.
We follow the strategy of proof of Theorem 2.1 in [19] and Theorem 1.3 in [25]. Recall that these results do not apply directly because of the form of our mixing.
It is easy to see that

$$\begin{aligned}
\sqrt{\frac{n}{\ell}}(\hat{B}^2_{n,2}-\mathbb{E}(\frac{S(\ell)}{\sqrt{\ell}})^2) &= \sqrt{\frac{n}{\ell}}\frac{1}{(n-\ell+1)}\sum_{j=0}^{n-\ell}[(\frac{S_j(\ell)}{\sqrt{\ell}})^2-\mathbb{E}(\frac{S_j(\ell)}{\sqrt{\ell}})^2]\\
&= \sqrt{\frac{n}{\ell}}\frac{1}{(n-\ell+1)}\sum_{m=0}^{k_n}\xi_{m,n}+\sqrt{\frac{n}{\ell}}\frac{1}{(n-\ell+1)}\sum_{m=0}^{k_n}\eta_{m,n}\\
&+\sqrt{\frac{n}{\ell}}\frac{1}{(n-\ell+1)}\sum_{j=(k_n+1)(2\ell+r)}^{n-\ell}[(\frac{S_j(\ell)}{\sqrt{\ell}})^2-\mathbb{E}(\frac{S_j(\ell)}{\sqrt{\ell}})^2]\\
&= J_{1,n}+J_{2,n}+J_{3,n}.
\end{aligned}$$

By Lemma 4.2 we have

$$\begin{aligned}
\mathrm{Var}(J_{3,n}) &= \frac{n}{\ell}\frac{1}{(n-\ell+1)^2}\mathrm{Var}[\sum_{j=(k_n+1)(2\ell+r)}^{n-\ell}((\frac{S_j(\ell)}{\sqrt{\ell}})^2-\mathbb{E}(\frac{S_j(\ell)}{\sqrt{\ell}})^2)]\\
&\leq \frac{2A_2'n(2\ell+r)}{(n-\ell+1)^2}(C(\ell,\ell)+8)||(\frac{S(\ell)}{\sqrt{\ell}})^2||^2_{2+\delta}\sum_{r=1}^{[\frac{n}{\ell}]+1}\alpha^*(r)^{\frac{\delta}{\delta+2}},
\end{aligned}$$

where A_2' is a constant. Also by Lemma 4.2 as $k_n\leq(n/r)$ we have

$$\begin{aligned}
\mathrm{Var}(J_{2,n}) &= \frac{n}{(n-\ell+1)^2\ell}\mathrm{Var}(\sum_{m=0}^{k_n}\eta_{m,n})\\
&\leq \frac{2\ell n^2A_3'(C(\ell,\ell)+8)}{r(n-\ell+1)^2}||(\frac{S(\ell)}{\sqrt{\ell}})^2||^2_{2+\delta}\sum_{r=1}^{[\frac{n}{\ell}]+1}\alpha^*(r)^{\frac{\delta}{\delta+2}},
\end{aligned} \tag{4.27}$$

where $A_3^{'}$ is a constant. Consider $r = \sqrt{n}$, and $\ell = \lceil \ln(n)/(5\ln K)\rceil + x$, then $\mathrm{Var}(J_{2,n})$ and $\mathrm{Var}(J_{3,n})$ go to 0.
Therefore to complete the proof of Proposition 4.7 it remains to show that

$$J_{1,n} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{4\sigma^4}{3}), \tag{4.28}$$

or equivalently,

$$\frac{1}{\sqrt{n\ell}} \sum_{m=0}^{k_n} \xi_{m,n} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{4\sigma^4}{3}). \tag{4.29}$$

We have

$$\frac{1}{\sqrt{n\ell}} \sum_{m=0}^{k_n} \xi_{m,n} = \sqrt{\frac{r}{n}}\sqrt{\frac{1}{r\ell}} \sum_{m=0}^{k_n} \xi_{m,n} = \sqrt{\frac{r}{n}} \sum_{m=0}^{k_n} \zeta_{m,n}, \tag{4.30}$$

where we denote by $\zeta_{m,n} = \sqrt{\frac{1}{r\ell}}\xi_{m,n}$. Recall that by (3.11) and $\mathbb{E}(|X_i|^{4+3\delta}) < \infty$,

$$\left|S_j(\ell)/\sqrt{\ell}\right|^{4+2\delta} \text{ is U.I.} \tag{4.31}$$

In addition with our choice of ℓ and r we have

$$(\frac{r}{\ell})^{1+\frac{\delta}{2}}\mathbb{E}|(\frac{S(\ell)}{\sqrt{\ell}})^2|^{2+\delta} I((\frac{S(\ell)}{\sqrt{\ell}})^2 \geq \frac{1}{2}\sqrt{\frac{r}{\ell}}) = o(1). \tag{4.32}$$

By stationarity and (4.32) we have

$$\begin{aligned}\sup_m \mathbb{E}|\zeta_{m,n}|^{2+\delta} I(|\zeta_{m,n}| > \frac{r}{\ell}) &\leq 2^{\delta+1}(\frac{r}{\ell})^{1+\frac{\delta}{2}}\mathbb{E}|(\frac{S(\ell)}{\sqrt{\ell}})^2|^{2+\delta} I((\frac{S(\ell)}{\sqrt{\ell}})^2 \geq \frac{1}{2}\sqrt{\frac{r}{\ell}}) \\ &= o(1) \ \ as\ n \to \infty.\end{aligned} \tag{4.33}$$

which means that $\left\{|\zeta_{m,n}|^{2+\delta}\right\}$ is uniformly integrable. Now we will prove that

$$\mathrm{Var}(\sqrt{\frac{r}{n}} \sum_{m=0}^{k_n} \zeta_{m,n}) \to \frac{4\sigma^4}{3}. \tag{4.34}$$

We have

$$\mathrm{Var}(\sqrt{\frac{r}{n}} \sum_{m=0}^{k_n} \zeta_{m,n}) = \frac{r}{n} \sum_{m=0}^{k_n} \mathbb{E}(\zeta_{m,n}^2) + 2 \sum_{i=0}^{k_n-1} \sum_{j=i+1}^{k_n} \mathbb{E}(\sqrt{\frac{r}{n}}\zeta_{j,n}\sqrt{\frac{r}{n}}\zeta_{i,n}).$$

Using (2.1) and with our choices of $\ell = [\ln(n)/5\ln(K)] + x$ and $r = \sqrt{n}$ we have $\forall\ j > i$

$$\begin{aligned}\mathbb{E}(\sqrt{\frac{r}{n}}\zeta_{i,n}\sqrt{\frac{r}{n}}\zeta_{j,n}) &= \frac{1}{n\ell}\mathbb{E}(\xi_{i,n}\xi_{j,n}) = \frac{1}{n\ell^3}\mathrm{Cov}(\sum_{k=i(2\ell+r)}^{i(2\ell+r)+r-1} S_k^2(\ell), \sum_{s=j(2\ell+r)}^{j(2\ell+r)+r-1} S_s^2(\ell)) \\ &\leq \frac{A_4^{'}}{n\ell^3} \sum_{k=i(2\ell+r)}^{i(2\ell+r)+r-1} \sum_{s=j(2\ell+r)}^{j(2\ell+r)+r-1} (C(\ell,\ell)+8)\left\|S^2(\ell)\right\|_{2+\delta}^2 \alpha^*(s-k)^{\frac{\delta}{\delta+2}} \\ &\leq \frac{A_4^{'}\left(K^{2x}n^{\frac{2}{5}}+8\right)r^2}{n\ell^3}\left\|S^2(\ell)\right\|_{2+\delta}^2 \alpha^*\left((j-i+1)(2\ell+r)\right)^{\frac{\delta}{\delta+2}} \\ &\leq \frac{A_4^{'}\left(n^{\frac{2}{5}}+8\right)}{\ell^3}\left\|S^2(\ell)\right\|_{2+\delta}^2 \alpha^*\left((j-i+1)(2\ell+r)\right)^{\frac{\delta}{\delta+2}},\end{aligned} \tag{4.35}$$

where $A_4^{'}$ is a constant. Using Theorem 2.6 to bound $\|S^2(\ell)\|_{2+\delta}^2$ and with our hypothesis on the coefficients α^* (specifically, the assumption that
$a > ((4+3\delta)(\delta+2))/\delta)$, we get:

$$\sum_{i=0}^{k_n-1} \sum_{j=i+1}^{k_n} \mathbb{E}(\sqrt{\frac{r}{n}}\zeta_{i,n}\sqrt{\frac{r}{n}}\zeta_{j,n}) = o(1). \tag{4.36}$$

So that, it remains to control $\frac{r}{n}\sum_{m=0}^{k_n}\mathbb{E}(\zeta_{m,n}^2) = \frac{r}{n}\sum_{m=0}^{k_n}\mathbb{E}(\zeta_{0,n}^2)$. By our choices of ℓ and r it remains that as $n\to\infty$, $\frac{r}{n}\sum_{m=0}^{k_n}\mathbb{E}(\zeta_{0,n}^2) \sim \mathbb{E}(\zeta_{0,n}^2)$.
We have

$$\begin{aligned}
\mathbb{E}(\zeta_{0,n}^2) &= \mathbb{E}[\frac{1}{\sqrt{r\ell}}\sum_{j=0}^{r-1}\frac{S_j^2(\ell)}{\sqrt{\ell}} - \mathbb{E}(\frac{S_j^2(\ell)}{\sqrt{\ell}})]^2 \\
&= \frac{1}{r\ell^3}\sum_{j=0}^{r-1}\text{Var}(S_j^2(\ell)) + \frac{2}{r\ell^3}\sum_{j=0}^{r-2}\sum_{i=j+1}^{r-1}\text{Cov}(S_j^2(\ell),S_i^2(\ell)) \\
&= \underbrace{\frac{1}{\ell^3}\text{Var}(S^2(\ell))}_{T_{1,n}} + \underbrace{\frac{2}{r\ell^3}\sum_{j=r-\ell}^{r-2}\sum_{i=j+1}^{r-1}\text{Cov}(S_j^2(\ell),S_i^2(\ell))}_{T_{2,n}} + \\
&\underbrace{\frac{2}{r\ell^3}\sum_{j=0}^{r-\ell-1}\sum_{i=j+1}^{j+\ell-1}\text{Cov}(S_j^2(\ell),S_i^2(\ell))}_{T_{3,n}} + \underbrace{\frac{2}{r\ell^3}\sum_{j=0}^{r-\ell-1}\sum_{i=j+\ell}^{r-1}\text{Cov}(S_j^2(\ell),S_i^2(\ell))}_{T_{4,n}} \\
&= T_{1,n}+T_{2,n}+T_{3,n}+T_{4,n}.
\end{aligned} \tag{4.37}$$

With our choices of ℓ and r and the hypothesis on the mixing coefficients, we show that $T_{1,n}$, $T_{2,n}$ and $T_{4,n}$ go to 0 as $n\to\infty$. Also, we have

$$T_{3,n} \sim 2\frac{1}{\ell}\sum_{i=1}^{\ell-1} Cov\left(\left(\frac{S(\ell)}{\sqrt{\ell}}\right)^2, \left(\frac{S_i(\ell)}{\sqrt{\ell}}\right)^2\right).$$

At this stage, we need the following lemma which is another writing of Lemma 2.5 of [25] which itself is derived by a careful analysis of Theorem 1.2 in [19].

Lemma 4.8. *Let $(X_n)_{n\geq 1}$ be a stationary $\alpha^*_{(v,v)}-$ mixing sequences for all $(u,v)\in\mathbb{N}^*$ with $\mathbb{E}(X_1)=0$. Assume that (3.3) and (3.4) hold, then*

$$\begin{aligned}
\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{i=1}^{\ell-1} Cov((\frac{S(\ell)}{\sqrt{\ell}})^2,(\frac{S_i(\ell)}{\sqrt{\ell}})^2) &= \int_0^1 Cov(\sigma^2 W(1)^2, \sigma^2(W(1+t)-W(t))^2)dt \\
&= \frac{2\sigma^4}{3},
\end{aligned} \tag{4.38}$$

where $(W(t), 0\leq t\leq 1)$ is a standard wiener process and Var$(S(\ell))/\sqrt{\ell}\to\sigma^2$ as $\ell\to\infty$.

Remark 4.9. *We note that Lemma 4.8 holds by Theorem 3.4 and by the uniform integrability of $(S(\ell)/\sqrt{\ell})^2, \ell\geq 1$.*

Lemma 4.8 leads to

$$\lim_{\ell\to\infty} 2\frac{1}{\ell}\sum_{i=1}^{\ell-1}\text{Cov}((\frac{S(\ell)}{\sqrt{\ell}})^2,(\frac{S_i(\ell)}{\sqrt{\ell}})^2) = \frac{4\sigma^4}{3}. \tag{4.39}$$

Finally as it was proved in [25] we have

$$\frac{\widehat{B}_{n,2} - \mathbb{E}(|\frac{S(\ell)-\ell\mu}{\sqrt{\ell}}|^2)^{\frac{1}{2}}}{\widehat{B}^2_{n,2} - \mathbb{E}((\frac{S(\ell)-\ell\mu}{\sqrt{\ell}})^2)} \to \frac{1}{2\sigma} \ \text{ in probability.} \tag{4.40}$$

and by (4.39) we get

$$\frac{n}{\ell}\mathrm{Var}(\widehat{B}_{n,2} - \mathbb{E}(\frac{S(\ell)-\ell\mu}{\sqrt{\ell}}|^2)^{\frac{1}{2}}) \to \frac{\sigma^2}{3}. \tag{4.41}$$

Finally Proposition 4.7 is a consequence of (4.41) and of (4.16). □

Remark 4.10. *Note that if $C(u,v) \leq K(u+v)^{\beta}$ i.e. $C(u,v)$ is polynomial in u and v and if $\ell = n^s$ then by the same reasoning (4.36) holds.*

Proof of Theorem 3.7. Theorem 3.7 is a corollary of Proposition 4.7. Without loss of generality, assume that $\mu = 0$. Recall that

$$\overline{X}_{\ell} = \frac{1}{(n-\ell+1)} \sum_{j=0}^{n-\ell} S_j(\ell).$$

We remark that since $a > \dfrac{(4+3\delta)(2+\delta)}{\delta}$, Remark 2.8 implies that

$$\left\{ \left(\frac{S_n - n\mu}{\sqrt{n}} \right)^{4+2\delta}, \ n \geq 1 \right\},$$

is uniformly integrable. To conclude the proof of the theorem, it suffices to prove that

$$\sqrt{\frac{n}{\ell}}\mathbb{E}\left|\widehat{B}^2_{n,2} - B^2_{n,2}\right| \to 0, \text{ as n goes to infinity,}$$

which can be done by following [25] and [19]. □

5 Some examples

In [15], we have given some examples of conditionally independent r.v.s $(X_i)_{(i\in N)}$ given a factor $(V_1,\ldots,V_n)$ that are relevant from the risk theory point of view. In fact, we considered that the structure of dependence between r.v.s $(X_i)_{(i\in N)}$ may come from a time-varying common factor which represents the evolution of socio-economic and natural environment. For these examples we have considered that r.v.s X_n, $n \geq 1$, are controlled by an unbounded memory of the factor. It should be noted that by unbounded memory of the factor we mean that the conditional independence is with respect to a length varying factor vector.

We have proved in [15] that these example satisfy the $\alpha^*_{(u,v)}-$ mixing property. We recall these examples here for completeness. Also, we have noticed that for these example the coefficients $C(u,v)$ are exponential in (u,v), so that we cannot obtain the mixing property with the classical α or Φ-mixing coefficients.

5.1 A discrete example

Consider the process $(I_i)_{(i\in N)}$ such that I_i's are Bernoulli r.v's conditionally to $\underline{V}_i = (V_1\ldots,V_i)$ and conditionally independent with respect to $\underline{V}_i$ where $(V_i)_{(i\in N)}$ is a mixing sequence of independent and identically distributed Bernoulli r.v.s with parameter q. We shall assume that the conditional law of I_i has the following structure:

$$\mathbb{P}(I_i = 1|\underline{V}_i) = K\sum_{j=1}^{i}\frac{(1+V_j)}{2^{i-j}},$$

where K is a constant of normalization. This example is inspired from insurance risk theory: $(I_i)_{(i\in N)}$ may modulate the frequency claim processes such that if $I_i = 1$ then there is a claim. On may consider the process $(X_i)_{(i\in N)}$, modeling individual claim amounts in non-live insurance for example, such that $X_i = I_i \times B_i$, where

- the I_i's are Bernoulli r.v's, conditionally independent with respect to $\underline{V}_i$, as above,
- the claim amount B_i's are considered independent and independent of the I_i's and of $\underline{V}_i$.
- $(V_i)_{(i\in N)}$ is a mixing sequence of Bernoulli random variables.

It has been shown in [15] that if the process $(V_i)_{i\in\mathbb{N}}$ is $\alpha^*_{(u,v)}$-mixing then so is $(X_n)_{n\in\mathbb{N}}$. Moreover, if the mixing coefficients of $(V_i)_{i\in\mathbb{N}}$ are denoted $\alpha^*_V(r)$ and $C_V(u,v)$ then, there exists $A > 1$, $B > 1$ such that the mixing coefficients of $(X_n)_{n\in\mathbb{N}}$ are such that

$$\alpha^*_X(r) \le 2^{-\frac{r}{2}} + \alpha^*_V(\frac{\alpha}{2}) \text{ and } C_X(u,v) \le \max(2A^v, B^{u+v}C_V(u,v)).$$

5.2 Example in an absolutely continuous case

Let the sequence of r.v.s $(X_i)_{(i\in N)}$ be such that for all $i \in \mathbb{N}$, X_i are conditionally independent with respect to the vector of the factor $\underline{V}_i$. Consider that $(X_i)_{(i\in N)}$ are Pareto r.v.s and that the sequence $(V_i)_{(i\in N)}$ is a sequence of i.i.d Bernoulli r.v.s with parameter q. In this case, the conditional law of X_i given $\underline{V}_i$ is $Pareto(\alpha,\theta_i)$ where $\alpha > 2$ is the shape parameter and $\theta_i > 0$ is the scale parameter. We assume that for all $i \in \mathbb{N}$, the conditional density of X_i given $\underline{V}_i$ has the form

$$f^i_{\underline{V}_i}(x_i;\alpha,\theta_i) = \alpha \times \frac{\theta_i^\alpha}{x_i^{\alpha+1}} \text{ for } x_i \ge \theta_i,$$

where the scale parameter θ_i is a r.v. depending on $\underline{V}_i$ and on α such that

$$\theta_i^\alpha = K\sum_{j=1}^{i}\frac{1+V_j}{2^{i-j}}.$$

It has been shown in [15] that if the process $(V_i)_{i\in\mathbb{N}}$ is $\alpha^*_{(u,v)}$-mixing then so is $(X_n)_{n\in\mathbb{N}}$. As in the previous example, explicit bounds on the mixing coefficients of $(X_n)_{n\in\mathbb{N}}$, depending on those of $(V_i)_{i\in\mathbb{N}}$ may be obtained.

Concluding remarks

Our results (Theorems 3.5 and 3.7) apply for sequences of stationary r.v.s. Below, we mention that if a mixing sequence is converging toward a stationary state, then the limit process is also mixing.

Assume that $(X_n)_{(n\in\mathbb{N})}$ is a sequence of asymptotically stationary r.v.s. That is, there is a stationary sequence $(Y_n)_{(n\in\mathbb{N})}$ such that for all bounded function $g:\mathbb{R}^k \to \mathbb{R}$ and $n \in \mathbb{N}$ we have for all $k \in \mathbb{N}$,

$$\mathbb{E}(g(X_n,\dots,X_{n+k})) \overset{n\to\infty}{\to} \mathbb{E}(g(Y_0,\dots,Y_k)). \tag{5.1}$$

Proposition 5.1. *Assume that $(X_n)_{n\in\mathbb{N}}$ is $\alpha^*_{(u\,v)}-$ mixing and $(Y_n)_{(n\in\mathbb{N})}$ is a stationary process satisfying (5.1). Then $(Y_n)_{(n\in\mathbb{N})}$ satisfy the $\alpha^*_{(u\,v)}-$ mixing property with the same coefficients.*

Proposition 5.1 implies if $(X_n)_{(n\in\mathbb{N})}$ is a $\alpha^*_{(u\,v)}-$ mixing and asymptotically stationary sequence then our results apply to the limit process. Examples considered above are asymptotically stationary.

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